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NONLINEAR ANALYSIS OF CROSS-PLY THICK CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION

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Abstract—Buckling, postbuckling, nonlinear vibration and parametric resonance of thick circular cylindrical shells under axial compression are analyzed in this paper. The theory developed is based on a nonlinear and non-shallow thick shell theory, with its final equations involving two unknowns, the circumferential displacement v and the radial displacement w. The shell wall is cross-ply laminated. The plies are specially orthotropic, but the lamination can be unsymmetric. The axial load is assumed to be harmonically time dependent, or constant as a special case. The governing nonlinear partial differential equations are reduced to nonlinear ordinary differential equations in terms of time by the Galerkin procedure. Then, an asymptotic method is used to solve the resulting nonlinear ordinary differential equations. The numerical results for buckling loads are shown to compare very well with those of three-dimensional theories in the literature, even for very thick shells. The effects of lay-up and thickness on postbuckling equilibrium, nonlinear vibration and parametric resonance are demonstrated by examples. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Buckling and postbuckling problems for cylindrical shells are usually tackled by thin-and-shallow shell theories, in which the radius of the shell is assumed to be much larger than its thickness and than the wavelengths of the buckling wave. However, it was shown [e.g. by Kardomateas (1995)], that such theories do not give good results for thick shells. This conclusion applies to both shallow and non-shallow thin shell theories. For laminated composite shells, the thickness/radius ratio is usually larger than that of metal ones. Moreover, the lay-ups of loaded laminates may play a greater role for thick shells than for thin ones. Therefore, there is a great need for thick shell theories for the analysis of axially compressed laminated cylindrical shells.

The literature contains relatively few papers related to the problems addressed in this paper. Recent such papers include the following. Bert and Birman (1988) gave a good account of the dynamic stability of thick cylindrical shells under harmonically time dependent axial compression. Anastasiadis *et al.* (1994), and Tabiei and Simitses (1994), used nonlinear thick-shell kinematic relations together with a high-order displacement mode to develop a set of governing equations for buckling problems of thick shells, then solved them for special cases individually. Xavier *et al.* (1995) dealt with the buckling and vibration problems of thick shells by using a layerwise theory and a high-order displacement mode. Finally, Kardomateas (1995), and Ye and Soldatos (1995), presented three-dimensional analyses for buckling problems of orthotropic and cross-ply thick shells.

Compared with the papers cited above, the main features of the present study include the following: (1) the dynamic case with harmonically time dependent axial compression is considered, so that nonlinear vibration and parametric resonance are covered; (2)

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nonlinear analysis is used so that initial postbuckling parameters, nonlinear frequencies and parametric resonance curves can be found; (3) a non-shallow thick shell theory is used, so that the analysis can give quite accurate results for very thick shells and is even good enough for thick cylinders buckling or vibrating as columns; (4) the inertia and nonlinear terms related to the circumferential displacement v are included in the kinetic equations, so that the final equations involve both circumferential and normal displacements, v and w, which appreciably improves the accuracy.

2. THEORY

A cylindrical coordinate system (ρ, θ, x) is used for the analysis, where ρ is the length coordinate in the radial direction, θ is the angular coordinate in circumferential direction, and x is the length coordinate in the axial direction. Whenever convenient, another right-hand coordinate system (x, y, z) is also used, where $y = R\theta$, $z = \rho - R$, and R is the radius of the shell.

The displacement field of the first-order shear-deformation theory assumes that

$$u_x = u + z\psi, \quad u_y = v + z\phi, \quad u_z = w \tag{1}$$

where $\{u_x, u_y, u_z\}$ is the displacement vector in the (x, y, z)-coordinate system at any point in the shell wall. The strains at this point can be expressed as

$$\varepsilon_x = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right)^2 \tag{2a}$$

$$\varepsilon_y = \frac{R}{\rho}\left(\frac{\partial u_y}{\partial y} + \frac{u_z}{R}\right) + \frac{1}{2}\left(\frac{R}{\rho}\right)^2 \left[\left(\frac{\partial u_z}{\partial y} - \frac{u_y}{R}\right)^2 + \left(\frac{\partial u_x}{\partial y}\right)^2\right] \tag{2b}$$

$$\gamma_{xy} = \frac{R}{\rho}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{R}{\rho}\frac{\partial u_z}{\partial x}\left(\frac{\partial u_z}{\partial y} - \frac{u_y}{R}\right) \tag{2c}$$

$$\gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} \tag{2d}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{R}{\rho}\left(\frac{\partial u_z}{\partial y} - \frac{u_y}{R}\right) + \frac{R}{\rho}\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z}. \tag{2e}$$

These expressions are the same as those used by Stein (1986), but with some simplification based on the assumption of small strains which gives

$$\frac{\partial u_x}{\partial x} \ll 1 \quad \text{and} \quad \frac{R}{\rho}\left(\frac{\partial u_y}{\partial y} + \frac{u_z}{R}\right) \ll 1. \tag{3}$$

Substituting eqn (1) into eqns (2a–e) to eliminate u_x, u_y and u_z and making the additional assumption that all nonlinear terms related to u, ψ and ϕ should be neglected gives

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \psi}{\partial x} \tag{4a}$$

$$\varepsilon_y = \frac{R}{\rho}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) + \frac{1}{2}\left(\frac{R}{\rho}\right)^2 \left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)^2 + z\frac{R}{\rho}\frac{\partial \phi}{\partial y} \tag{4b}$$

$$\gamma_{xy} = \frac{R}{\rho}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{R}{\rho}\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + z\left(\frac{R}{\rho}\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}\right) \tag{4c}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \psi \tag{4d}$$

$$\gamma_{yz} = \frac{R}{\rho}\left(\frac{\partial w}{\partial y} - \frac{v}{R} + \phi\right). \tag{4e}$$

Based on the kinematic eqns (4a–e), application of the principle of virtual work yields the equations of equilibrium

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_{yx}}{\partial y} + q_x = 0 \tag{5a}$$

$$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{Q_y}{R} + \frac{\partial}{\partial x}\left(T_x \frac{\partial v}{\partial x}\right) + \frac{T_y^*}{R}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + \frac{T_{yx}}{R}\frac{\partial w}{\partial x} + q_y = 0 \tag{5b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{T_y}{R} + \frac{\partial}{\partial x}\left(T_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left[T_y^*\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)\right]$$
$$+ \frac{\partial}{\partial x}\left[T_{yx}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right)\right] + \frac{\partial}{\partial y}\left(T_{yx}\frac{\partial w}{\partial x}\right) + q_z = 0 \tag{5c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x + m_x = 0 \tag{5d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + m_y = 0 \tag{5e}$$

and the natural boundary conditions

$$[(T_x + P)\delta u]|_0^L = 0 \tag{6a}$$

$$\left[\left(T_x \frac{\partial v}{\partial x} + T_{xy}\right)\delta v\right]_0^L = 0 \tag{6b}$$

$$\left[\left(T_x \frac{\partial w}{\partial x} + T_{yx}\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + Q_x\right)\delta w\right]_0^L = 0 \tag{6c}$$

$$(M_x \delta\psi)|_0^L = 0 \tag{6d}$$

$$(M_{xy}\delta\phi)|_0^L = 0 \tag{6e}$$

$$(T_{yx}\delta u)|_0^{2\pi} = 0 \tag{6f}$$

$$(T_y \delta v)|_0^{2\pi} = 0 \tag{6g}$$

$$\left[\left(T_y^*\left(\frac{\partial w}{\partial y} - \frac{v}{R}\right) + T_{yx}\frac{\partial w}{\partial x} + Q_y\right)\delta w\right]_0^{2\pi} = 0 \tag{6h}$$

$$(M_{yx}\delta\psi)|_0^{2\pi} = 0 \tag{6i}$$

$$(M_y \delta\phi)|_0^{2\pi} = 0 \tag{6j}$$

where (q_x, q_y, q_z) and (m_x, m_y) are external loads (including inertia forces) per unit area of the middle surface, and the stress resultants are defined by

$$T_x = \int \sigma_x \frac{\rho}{R} \mathrm{d}z, \quad M_x = \int \sigma_x \frac{\rho}{R} z \, \mathrm{d}z \tag{7a}$$

$$T_y = \int \sigma_y \, \mathrm{d}z, \quad M_y = \int \sigma_y z \, \mathrm{d}z, \quad T_y^* = \int \sigma_y \frac{R}{\rho} \mathrm{d}z \tag{7b}$$

$$T_{xy} = \int \tau_{xy} \frac{\rho}{R} \mathrm{d}z, \quad M_{xy} = \int \tau_{xy} \frac{\rho}{R} z \, \mathrm{d}z, \quad T_{yx} = \int \tau_{xy} \, \mathrm{d}z, \quad M_{yx} = \int \tau_{xy} z \, \mathrm{d}z \tag{7c}$$

$$Q_x = \int \tau_{xz} \frac{\rho}{R} \mathrm{d}z, \quad Q_y = \int \tau_{yz} \, \mathrm{d}z \tag{7d}$$

where the integrals are over the whole thickness. Here these stress resultants are defined differently from Stein (1986), in order to retain the usual meanings of the stress resultants and the conventional forms of the kinetic equations. For example, M_x in eqn (7a) represents the moment, per unit length of y, produced by the stress σ_x about the junction line of the cross-section with the middle surface. The equations of equilibrium in terms of these stress resultants are very similar to those for thin shells.

In the present case there is no external load except for the inertia forces. Furthermore, it is assumed that the inertias related to u, ψ and ϕ can be neglected and that, for an axially compressed shell, the nonlinear terms related to T_y^* and T_{yx} in eqns (5b–c) can also be neglected. With these assumptions, eqns (5a–e) reduce to their final form:

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_{yx}}{\partial y} = 0 \tag{8a}$$

$$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{Q_y}{R} + T_x \frac{\partial^2 v}{\partial x^2} = \mu \ddot{v} \tag{8b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{T_y}{R} + T_x \frac{\partial^2 w}{\partial x^2} = \mu \ddot{w} \tag{8c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \tag{8d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{8e}$$

where μ is the mass per unit area of the middle surface and the dots denote differentiation with respect to time.

The shell wall is assumed to be cross-ply laminated and the constitutive equations of each ply can be written in the matrix form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \tag{10}$$

substitution of eqns (9) and (10) into (7a–e) yields

$$T_x = \hat{A}_{11}\left(\frac{\partial u}{\partial x}+\tilde{\varepsilon}_x\right)+A_{12}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)+\bar{A}_{12}\tilde{\varepsilon}_y+\hat{B}_{11}\frac{\partial \psi}{\partial x}+B_{12}\frac{\partial \phi}{\partial y} \tag{11a}$$

$$T_y = A_{12}\left(\frac{\partial u}{\partial x}+\tilde{\varepsilon}_x\right)+\bar{A}_{22}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)+\bar{\bar{A}}_{22}\tilde{\varepsilon}_y+B_{12}\frac{\partial \psi}{\partial x}+\bar{B}_{22}\frac{\partial \phi}{\partial y} \tag{11b}$$

$$T_{xy} = A_{66}\left(\frac{\partial u}{\partial y}+\tilde{\varepsilon}_{xy}\right)+\hat{A}_{66}\frac{\partial v}{\partial x}+B_{66}\frac{\partial \psi}{\partial y}+\hat{B}_{66}\frac{\partial \phi}{\partial x} \tag{11c}$$

$$T_{yx} = \bar{A}_{66}\left(\frac{\partial u}{\partial y}+\tilde{\varepsilon}_{xy}\right)+A_{66}\frac{\partial v}{\partial x}+\bar{B}_{66}\frac{\partial \psi}{\partial y}+B_{66}\frac{\partial \phi}{\partial x} \tag{11d}$$

$$M_x = \hat{B}_{11}\left(\frac{\partial u}{\partial x}+\tilde{\varepsilon}_x\right)+B_{12}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)+\bar{B}_{12}\tilde{\varepsilon}_y+\hat{D}_{11}\frac{\partial \psi}{\partial x}+D_{12}\frac{\partial \phi}{\partial y} \tag{11e}$$

$$M_y = B_{12}\left(\frac{\partial u}{\partial x}+\tilde{\varepsilon}_x\right)+\bar{B}_{22}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right)+\bar{\bar{B}}_{22}\tilde{\varepsilon}_y+D_{12}\frac{\partial \psi}{\partial x}+\bar{D}_{22}\frac{\partial \phi}{\partial y} \tag{11f}$$

$$M_{xy} = B_{66}\left(\frac{\partial u}{\partial y}+\tilde{\varepsilon}_{xy}\right)+\hat{B}_{66}\frac{\partial v}{\partial x}+D_{66}\frac{\partial \psi}{\partial y}+\hat{D}_{66}\frac{\partial \phi}{\partial x} \tag{11g}$$

$$M_{yx} = \bar{B}_{66}\left(\frac{\partial u}{\partial y}+\tilde{\varepsilon}_{xy}\right)+B_{66}\frac{\partial v}{\partial x}+\bar{D}_{66}\frac{\partial \psi}{\partial y}+D_{66}\frac{\partial \phi}{\partial x} \tag{11h}$$

$$Q_x = \hat{A}_{55}\left(\frac{\partial w}{\partial x}+\psi\right) \tag{11i}$$

$$Q_y = \bar{A}_{44}\left(\frac{\partial w}{\partial y}-\frac{v}{R}+\phi\right) \tag{11j}$$

where, with i and j each taking the appropriate one of the numbers 1, 2, 4, 5 and 6

$$\tilde{\varepsilon}_x = \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right],\quad \tilde{\varepsilon}_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{v}{R}\right)^2,\quad \tilde{\varepsilon}_{xy} = \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y}-\frac{v}{R}\right)$$

$$(A_{ij}, B_{ij}, D_{ij}) = \int Q_{ij}(1, z, z^2)\,\mathrm{d}z \tag{12a}$$

$$(\hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}) = \int Q_{ij}(1, z, z^2)\frac{\rho}{R}\,\mathrm{d}z \tag{12b}$$

$$(\bar{A}_{ij}, \bar{B}_{ij}, \bar{D}_{ij}) = \int Q_{ij}(1, z, z^2)\frac{R}{\rho}\,\mathrm{d}z \tag{12c}$$

$$(\bar{\bar{A}}_{ij}, \bar{\bar{B}}_{ij}, \bar{\bar{D}}_{ij}) = \int Q_{ij}(1, z, z^2)\left(\frac{R}{\rho}\right)^2\mathrm{d}z \tag{12d}$$

and the integrals are over the whole thickness. From eqns (12a–d) it can be seen that, in addition to the usual stiffnesses A_{ij}, B_{ij} and D_{ij}, the new stiffnesses, $\hat{A}_{ij}$, $\hat{B}_{ij}$, ..., $\bar{\bar{D}}_{ij}$, have been introduced for thick shells. They can be called modified stiffnesses. The following equations are used to calculate them.

$$(\hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}) = (A_{ij}, B_{ij}, D_{ij})+(B_{ij}, D_{ij}, H_{ij})/R,\quad H_{ij} = \int Q_{ij}z^3\,\mathrm{d}z$$

$$\bar{A}_{ij} = R\sum_{k=1}^{n} Q_{ij}^{k}\ln\frac{R+z_{k+1}}{R+z_k},\quad \bar{B}_{ij} = R(A_{ij}-\bar{A}_{ij}),\quad \bar{D}_{ij} = R(B_{ij}-\bar{B}_{ij})$$

$$\bar{\bar{A}}_{ij} = R^2 \sum_{k=1}^{n} Q_{ij}^k \left(\frac{1}{R+z_k} - \frac{1}{R+z_{k+1}}\right)$$

where n is the total number of the layers of the laminate, Q_{ij}^k are the material constants of the kth layer, and z_k is the thickness of the kth layer. During this computation, a so-called transverse shear correction factor, 5/6, should be introduced into the transverse shear stiffnesses. Equations (8a–e) and (11a–j) constitute the governing equations for this paper.

3. REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

If the stress resultants of eqns (11a–j) are used, eqns (8a–e) can be expressed in terms of the five unknowns, u, v, w, ψ and ϕ, but these are inconvenient when solving and so they will be replaced by alternative unknown functions as now described. For simplicity, attention is confined to simply-supported boundary conditions at the two ends, i.e.

$$T_x(0,y) = T_x(L,y) = -P(t) = -(P_0 + P_t \cos ft) \tag{13a}$$

$$M_x(0,y) = M_x(L,y) = 0 \tag{13b}$$

$$v(0,y) = v(L,y) = 0 \tag{13c}$$

$$w(0,y) = w(L,y) = 0 \tag{13d}$$

$$\phi(0,y) = \phi(L,y) = 0 \tag{13e}$$

where $P(t)$ is the axial compression, and P_0, P_t and f are constants. It is obvious that these boundary conditions satisfy the natural boundary conditions of eqns (6a–e). The other conditions of eqns (6f–j) can easily be satisfied if the quantities appearing in these conditions are assumed to be periodic in θ with a period 2π. The conditions of eqns (13a–e) suggest that T_x, M_x, v, w and ϕ may be a better set of basic unknowns. By using eqns (11a) and (11f), the displacement u and the rotation ψ can be expressed in terms of these basic unknowns as

$$\frac{\partial u}{\partial x} = c_{11}T_x + c_{12}M_x - c_{13}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) - c_{14}\frac{\partial \phi}{\partial y} - c_{15}\tilde{\varepsilon}_x - c_{16}\tilde{\varepsilon}_y \tag{14a}$$

$$\frac{\partial \psi}{\partial x} = c_{12}T_x + c_{22}M_x - c_{23}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) - c_{24}\frac{\partial \phi}{\partial y} - c_{25}\tilde{\varepsilon}_x - c_{26}\tilde{\varepsilon}_y \tag{14b}$$

where the c_{ij} are constants derived from the physical constants A_{12}, etc, appearing in eqns (11a) and (11f).

In order to solve for the basic unknowns, it is assumed in eqns (8a–e) that

$$T_x = \sum_{m,n=1}^{N} T_{mn}(t) \sin \alpha_m x \cos n\theta - P(t) \tag{15a}$$

$$M_x = \sum_{m,n=1}^{N} H_{mn}(t) \sin \alpha_m x \cos n\theta \tag{15b}$$

$$v = V(t) \sin \alpha_p x \sin q\theta \tag{15c}$$

$$w = W(t) \sin \alpha_p x \cos q\theta \tag{15d}$$

$$\phi = \Phi(t) \sin \alpha_p x \sin q\theta \tag{15e}$$

where the T_{mn}, H_{mn}, V, W and Φ are to be solved for and $\alpha_i = i\pi/L$ for $i = m$ or p. The functions of eqns (15a–e) obviously satisfy eqns (13a–e). Substituting eqns (11a–j), (14a,

b) and (15a–e) into eqns (8a) and (8d) and then executing the Galerkin procedure gives, for $i, j = 1, 2, \ldots, N$,

$$Z_{ij}^{tt}T_{ij}+Z_{ij}^{th}H_{ij}+X_{ij}^{tv}V+X_{ij}^{tw}W+X_{ij}^{t\phi}\Phi+Y_{ij}^{tv}V^2+Y_{ij}^{tc}VW+Y_{ij}^{tw}W^2=0 \qquad (16a)$$

$$Z_{ij}^{ht}T_{ij}+Z_{ij}^{hh}H_{ij}+X_{ij}^{hv}V+X_{ij}^{hw}W+X_{ij}^{h\phi}\Phi+Y_{ij}^{hv}V^2+Y_{ij}^{hc}VW+Y_{ij}^{hw}W^2=0 \qquad (16b)$$

where

$$Z_{ij}^{tt}=\alpha_i^2+(\bar{A}_{66}c_{11}+\bar{B}_{66}c_{21})\beta_j^2,\quad Z_{ij}^{th}=(\bar{A}_{66}c_{12}+\bar{B}_{66}c_{22})\beta_j^2$$

$$X_{ij}^{tv}=[A_{66}\alpha_p^2\beta_q-(\bar{A}_{66}c_{13}+\bar{B}_{66}c_{23})\beta_q^3]\delta_{ip}\delta_{jq}$$

$$X_{ij}^{tw}=-(\bar{A}_{66}c_{13}+\bar{B}_{66}c_{23})\frac{\beta_q^2}{R}\delta_{ip}\delta_{jq}$$

$$X_{ij}^{t\phi}=[B_{66}\alpha_p^2-(\bar{A}_{66}c_{14}+\bar{B}_{66}c_{24})\beta_q^2]\beta_q\delta_{ip}\delta_{jq}$$

$$Y_{ij}^{tv}=(\bar{A}_{66}c_{15}+\bar{B}_{66}c_{25})\alpha_p^2\beta_q^2\zeta_{i,2p}^{c}\delta_{j,2q}+(\bar{A}_{66}c_{16}+\bar{B}_{66}c_{26})\frac{\beta_q^2}{R^2}\zeta_{i,2p}^{s}\delta_{j,2q}$$

$$Y_{ij}^{tc}=(\bar{A}_{66}c_{16}+\bar{B}_{66}c_{26})\frac{2\beta_q^3}{R}\zeta_{i,2p}^{s}\delta_{j,2q}+\bar{A}_{66}\frac{\alpha_p^2\beta_q}{R}\eta_{i,2p}\delta_{j,2q}$$

$$\begin{aligned}Y_{ij}^{tw}=&-(\bar{A}_{66}c_{15}+\bar{B}_{66}c_{25})\alpha_p^2\beta_q^2\zeta_{i,2p}^{c}\delta_{j,2q}\\&+(\bar{A}_{66}c_{16}+\bar{B}_{66}c_{26})\beta_q^4\zeta_{i,2p}^{s}\delta_{j,2q}+\bar{A}_{66}\alpha_p^2\beta_q^2\eta_{i,2p}\delta_{j,2q}\end{aligned}$$

$$Z_{ij}^{ht}=(\bar{B}_{66}c_{11}+\bar{D}_{66}c_{21})\beta_j^2+\hat{A}_{55}c_{21},\quad Z_{ij}^{hh}=\alpha_i^2+(\bar{B}_{66}c_{12}+\bar{D}_{66}c_{22})\beta_j^2+\hat{A}_{55}c_{22}$$

$$X_{ij}^{hv}=[B_{66}\alpha_p^2-(\bar{B}_{66}c_{13}+\bar{D}_{66}c_{23})\beta_q^2-\hat{A}_{55}c_{23}]\beta_q\delta_{ip}\delta_{jq}$$

$$X_{ij}^{hw}=-[(\bar{B}_{66}c_{13}+\bar{D}_{66}c_{23})\beta_q^2/R+\hat{A}_{55}(\alpha_p^2+c_{23}/R)]\delta_{ip}\delta_{jq}$$

$$X_{ij}^{h\phi}=[D_{66}\alpha_p^2-(\bar{B}_{66}c_{14}+\bar{D}_{66}c_{24})\beta_q^2-\hat{A}_{55}c_{24}]\beta_q\delta_{ip}\delta_{jq}$$

$$\begin{aligned}Y_{ij}^{hv}=&[(\bar{B}_{66}c_{15}+\bar{D}_{66}c_{25})\beta_q^2+\hat{A}_{55}c_{25}/4]\alpha_p^2\zeta_{i,2p}^{c}\delta_{j,2q}\\&+[(\bar{B}_{66}c_{16}+\bar{D}_{66}c_{26})\beta_q^2+\hat{A}_{55}c_{26}/4]\frac{1}{R^2}\zeta_{i,2p}^{s}\delta_{j,2q}\end{aligned}$$

$$Y_{ij}^{hc}=[(\bar{B}_{66}c_{16}+\bar{D}_{66}c_{26})\beta_q^2+\hat{A}_{55}c_{26}/4]\frac{2\beta_q}{R}\zeta_{i,2p}^{s}\delta_{j,2q}+\bar{A}_{66}\frac{\alpha_p^2\beta_q}{R}\eta_{i,2p}\delta_{j,2q}$$

$$\begin{aligned}Y_{ij}^{hw}=&-[(\bar{B}_{66}c_{15}+\bar{D}_{66}c_{25})\beta_q^2+\hat{A}_{55}c_{25}/4]\alpha_p^2\zeta_{i,2p}^{c}\delta_{j,2q}\\&+[(\bar{B}_{66}c_{16}+\bar{D}_{66}c_{26})\beta_q^2+\hat{A}_{55}c_{26}/4]\beta_q^2\zeta_{i,2p}^{s}\delta_{j,2q}+\bar{A}_{66}\alpha_p^2\beta_q^2\eta_{i,2p}\delta_{j,2q}.\end{aligned}$$

Here, δ_{ij} is the Kronecker delta and

$$\beta_q=q/R,\quad \eta_{ij}=\frac{2i[1-(-1)^{i+j}]}{(i^2-j^2)\pi}$$

$$\zeta_{ij}^{s}=(\eta_{i0}-\eta_{ij})/2,\quad \zeta_{ij}^{c}=(\eta_{i0}+\eta_{ij})/2.$$

The steps starting beneath eqn (15e) can be repeated for eqn (8e) to obtain

$$C_tT_{pq}+C_hH_{pq}+C_vV+C_wW=C_\phi\Phi \qquad (17)$$

where

$$C_t = [(B_{12}+B_{66})c_{11}+(D_{12}+D_{66})c_{21}]\beta_q$$

$$C_h = [(B_{12}+B_{66})c_{12}+(D_{12}+D_{66})c_{22}]\beta_q$$

$$C_v = \hat{B}_{66}\alpha_p^2+[\bar{B}_{22}-(B_{12}+B_{66})c_{13}-(D_{12}+D_{66})c_{23}]\beta_q^2-\bar{A}_{44}/R$$

$$C_w = \{[\bar{B}_{22}-(B_{12}+B_{66})c_{13}-(D_{12}+D_{66})c_{23}]/R-\bar{A}_{44}\}\beta_q$$

$$C_\phi = [(B_{12}+B_{66})c_{14}+(D_{12}+D_{66})c_{24}-\bar{D}_{22}]\beta_q^2-\hat{D}_{66}\alpha_p^2-\bar{A}_{44}.$$

The unknown function Φ in eqns (16a, b) can be eliminated by using eqn (17) to obtain

$$a_{ij}^{11}T_{ij}+a_{ij}^{12}H_{ij}+a_{ij}^{13}V+a_{ij}^{14}W+Y_{ij}^{tv}V^2+Y_{ij}^{tc}VW+Y_{ij}^{tw}W^2 = 0$$

$$a_{ij}^{21}T_{ij}+a_{ij}^{22}H_{ij}+a_{ij}^{23}V+a_{ij}^{24}W+Y_{ij}^{hv}V^2+Y_{ij}^{hc}VW+Y_{ij}^{hw}W^2 = 0 \quad i,j = 1,2,\ldots,N$$

where

$$a_{ij}^{11} = Z_{ij}^{tt}+\frac{C_t}{C_\phi}X_{ij}^{t\phi}, \quad a_{ij}^{12} = Z_{ij}^{th}+\frac{C_h}{C_\phi}X_{ij}^{t\phi}$$

$$a_{ij}^{13} = X_{ij}^{tv}+\frac{C_v}{C_\phi}X_{ij}^{t\phi}, \quad a_{ij}^{14} = X_{ij}^{tw}+\frac{C_w}{C_\phi}X_{ij}^{t\phi}$$

$$a_{ij}^{21} = Z_{ij}^{ht}+\frac{C_t}{C_\phi}X_{ij}^{h\phi}, \quad a_{ij}^{22} = Z_{ij}^{hh}+\frac{C_h}{C_\phi}X_{ij}^{h\phi}$$

$$a_{ij}^{23} = X_{ij}^{hv}+\frac{C_v}{C_\phi}X_{ij}^{h\phi}, \quad a_{ij}^{24} = X_{ij}^{hw}+\frac{C_w}{C_\phi}X_{ij}^{h\phi}.$$

The solution of these linear equations for T_{ij} and H_{ij} for each pair of (i,j), for $i, j = 1, 2, \ldots, N$, gives

$$T_{ij} = b_{ij}^{11}V+b_{ij}^{12}W+b_{ij}^{13}V^2+b_{ij}^{14}VW+b_{ij}^{15}W^2 \tag{18a}$$

$$H_{ij} = b_{ij}^{21}V+b_{ij}^{22}W+b_{ij}^{23}V^2+b_{ij}^{24}VW+b_{ij}^{25}W^2 \tag{18b}$$

where the coefficients $b_{ij}^{11},\ldots,b_{ij}^{25}$ can be determined from $a_{ij}^{11},\ldots,a_{ij}^{24}$ and $Y_{ij}^{tv},\ldots,Y_{ij}^{hw}$.

Application of the Galerkin procedure to eqns (8b, c) yields

$$d_t^v T_{pq}+d_h^v H_{pq}+(d_v^v-\alpha_p^2 P)V+d_w^v W+d_\phi^v\Phi-\frac{1}{2}\alpha_p^2 V\sum_{m,n=1}^{N} T_{mn}\zeta_{m,2p}^s\delta_{n,2q}+\mu\ddot{V} = 0 \tag{19a}$$

$$d_t^w T_{pq}+d_h^w H_{pq}+d_v^w V+(d_w^w-\alpha_p^2 P)W+d_\phi^w\Phi+\frac{1}{2}\alpha_p^2 W\sum_{m,n=1}^{N} T_{mn}\zeta_{m,2p}^s\delta_{n,2q}+\mu\ddot{W} = 0 \tag{19b}$$

where

$$d_t^v = [(A_{12}+A_{66})c_{11}+(B_{12}+B_{66})c_{21}]\beta_q, \quad d_h^v = [(A_{12}+A_{66})c_{12}+(B_{12}+B_{66})c_{22}]\beta_q$$

$$d_v^v = \hat{A}_{66}\alpha_p^2+[\bar{A}_{22}-(A_{12}+A_{66})c_{13}-(B_{12}+B_{66})c_{23}]\beta_q^2+\bar{A}_{44}/R^2$$

$$d_w^v = [\bar{A}_{22}+\bar{A}_{44}-(A_{12}+A_{66})c_{13}-(B_{12}+B_{66})c_{23}]\beta_q/R$$

$$d_\phi^v = \hat{B}_{66}\alpha_p^2+[\bar{B}_{22}-(A_{12}+A_{66})c_{14}-(B_{12}+B_{66})c_{24}]\beta_q^2-\bar{A}_{44}/R$$

$$d_t^w = (A_{12}c_{11}+B_{12}c_{21})/R-\hat{A}_{55}c_{21}, \quad d_h^w = (A_{12}c_{12}+B_{12}c_{22})/R-\hat{A}_{55}c_{22}$$

$$d_v^w = [(\bar{A}_{22}+\bar{A}_{44}-A_{12}c_{13}-B_{12}c_{23})/R+\hat{A}_{55}c_{23}]\beta_q$$

$$d_w^w = [(\bar{A}_{22}-A_{12}c_{13}-B_{12}c_{23})/R+\hat{A}_{55}c_{23}]/R+\bar{A}_{44}\beta_q^2+\hat{A}_{55}\alpha_p^2$$

$$d_\phi^w = [(\bar{B}_{22}-A_{12}c_{14}-B_{12}c_{24})/R+\hat{A}_{55}c_{24}-\bar{A}_{44}]\beta_q$$

After elimination of T_{ij} and H_{ij} by using eqns (18a, b), eqns (19a, b) become

$$\mu\ddot{V}+\left[f_t^v b_{pq}^{11}+f_h^v b_{pq}^{21}+d_v^v+\frac{C_v}{C_\phi}d_\phi^v-\alpha_p^2 P\right]V+\left[f_t^v b_{pq}^{12}+f_h^v b_{pq}^{22}+d_w^v+\frac{C_w}{C_\phi}d_\phi^v\right]W$$

$$+[f_t^v b_{pq}^{13}+f_h^v b_{pq}^{23}-e_{11}]V^2+[f_t^v b_{pq}^{14}+f_h^v b_{pq}^{24}-e_{12}]VW+[f_t^v b_{pq}^{15}+f_h^v b_{pq}^{25}]W^2$$

$$-e_{13}V^3-e_{14}V^2W-e_{15}VW^2=0 \quad (20a)$$

$$\mu\ddot{W}+\left[f_t^w b_{pq}^{11}+f_h^w b_{pq}^{21}+d_v^w+\frac{C_v}{C_\phi}d_\phi^w\right]V+\left[f_t^w b_{pq}^{12}+f_h^w b_{pq}^{22}+d_w^w+\frac{C_w}{C_\phi}d_\phi^w-\alpha_p^2 P\right]W$$

$$+[f_t^w b_{pq}^{13}+f_h^w b_{pq}^{23}]V^2+[f_t^w b_{pq}^{14}+f_h^w b_{pq}^{24}+e_{11}]VW+[f_t^w b_{pq}^{15}+f_h^w b_{pq}^{25}+e_{12}]W^2$$

$$+e_{13}V^2W+e_{14}VW^2+e_{15}W^3=0 \quad (20b)$$

where

$$e_{ij}=\frac{1}{2}\alpha_p^2\sum_{m,n=1}^{N}b_{mn}^{ij}\zeta_{m,2p}^{s}\delta_{n,2q}$$

$$e_{ij}=\frac{1}{2}\alpha_p^2\sum_{m,n=1}^{N}b_{mn}^{ij}\zeta_{m,2p}^{s}\delta_{n,2q}$$

$$f_t^v=d_t^v+\frac{C_t}{C_\phi}d_\phi^v,\quad f_h^v=d_h^v+\frac{C_h}{C_\phi}d_\phi^v,\quad f_t^w=d_t^w+\frac{C_t}{C_\phi}d_\phi^w,\quad f_h^w=d_h^w+\frac{C_h}{C_\phi}d_\phi^w.$$

Equations (20a, b) are the final nonlinear ordinary differential equations in terms of V and W used in the present paper.

4. INITIAL POSTBUCKLING

Consider a general nonlinear dynamic system of the form

$$\mu\ddot{u}_i+\sum_{j=1}^{N}[d_{ij}\dot{u}_j+(a_{ij}-\alpha^2P(t)\delta_{ij})u_j]+\sum_{j,k=1}^{N}b_{ijk}u_ju_k+\sum_{j,k,l=1}^{N}c_{ijkl}u_ju_ku_l=0,\quad i=1,2,\ldots,N \quad (21)$$

where the u_i, $i=1, 2, \ldots, N$, are the generalized displacements, $[a_{ij}]$ is a symmetric and positive definite matrix, $d_{ij}\dot{u}_j$ represents the equivalent viscous damping and the coefficients μ, α, d_{ij}, a_{ij} b_{ijk} and c_{ijkl} are all constants. It is obvious that eqns (20a, b) are a special case of eqn (21) for $N=2$ and $d_{ij}=0$. Let the matrix $[a_{ij}]$ have distinct eigenvalues

$$\lambda_1<\lambda_2<\cdots<\lambda_N$$

with eigenvectors

$$\{x_{1j}, x_{2j}, \ldots, x_{Nj}\}^{\mathrm{T}},\quad j=1,2,\ldots,N.$$

By introducing a new set of generalized displacements w_j such that

$$u_i=\sum_{j=1}^{N}x_{ij}w_j \quad (22)$$

eqn (21) can be transformed into

$$\mu\ddot{w}_m + (\lambda_m - \alpha^2 P(t))w_m + \sum_{n=1}^{N}\left(\sum_{i,j=1}^{N} x_{im}x_{jn}d_{ij}\right)\dot{w}_n + \sum_{n,s=1}^{N}\left(\sum_{i,j,k=1}^{N} x_{im}x_{jn}x_{ks}b_{ijk}\right)w_n w_s$$

$$+ \sum_{n,s,t=1}^{N}\left(\sum_{i,j,k,l=1}^{N} x_{im}x_{jn}x_{ks}x_{lt}c_{ijkl}\right)w_n w_s w_t = 0, \quad m = 1,2,\ldots,N. \quad (23)$$

(Note that w_m here and v_m below have no relation to the physical displacements w and v.)

Two special cases (i) and (ii) are now considered.

(i) *Linear static equilibrium.* In this case eqn (23) reduce to the system of equations of equilibrium

$$(\lambda_m - \alpha^2 P_0)w_m = 0, \quad m = 1,2,\ldots,N \quad (24)$$

where P_0 is the constant axial compression. From eqn (24) the buckling load can be obtained as

$$P_{cr} = \lambda_1/\alpha^2 \quad (25)$$

which is used to define a dimensionless load $\bar{P}_{cr}(t)$ as

$$\bar{P}_{cr}(t) = \frac{P(t)}{P_{cr}} = \frac{P_0}{P_{cr}} + \frac{P_t}{P_{cr}}\cos ft = \bar{P}_0 + \bar{P}_t\cos ft. \quad (26)$$

(ii) *Linear free vibration.* In this case eqn (23) becomes

$$\mu\ddot{w}_m + \lambda_m w_m = 0, \quad m = 1,2,\ldots,N \quad (27)$$

which give the natural frequencies ω_m as

$$\omega_m^2 = \lambda_m/\mu. \quad (28)$$

The lowest natural frequency ω_1 is used to define a dimensionless time $\bar{t}$ as

$$\bar{t} = \omega_1 t. \quad (29)$$

Introducing $\bar{P}(t)$, $\bar{t}$ and dimensionless generalized displacements $v_m = w_m/T$, where T is the thickness, into eqn (23) gives

$$v_m'' + \Omega_m^2 v_m = v_m\bar{P}_t\cos\bar{f}\bar{t} - \sum_{n=1}^{N}\bar{d}_{mn}v_n' - \sum_{n,s=1}^{N}\bar{b}_{mns}v_n v_s - \sum_{n,s,t=1}^{N}\bar{c}_{mnst}v_n v_s v_t, \quad m = 1,2,\ldots,N \quad (30)$$

where the prime denotes the derivative with respect to $\bar{t}$ and Ω_m^2, $\bar{d}_{mn}$, $\bar{b}_{mns}$ and $\bar{c}_{mnst}$ are the dimensionless quantities

$$\bar{f} = \frac{f}{\omega_1}$$

$$\Omega_m^2 = \frac{\lambda_m}{\lambda_1} - \bar{P}_0$$

$$\bar{d}_{mn} = \frac{T}{\lambda_1}\sum_{i,j=1}^{N} x_{im}x_{jn}d_{ij}$$

$$\bar{b}_{mns} = \frac{T^2}{\lambda_1} \sum_{i,j,k=1}^{N} x_{im} x_{jn} x_{ks} b_{ijk}$$

$$\bar{c}_{mnst} = \frac{T^3}{\lambda_1} \sum_{i,j,k,l=1}^{N} x_{im} x_{jn} x_{ks} x_{lt} c_{ijkl}.$$

The coefficients in eqn (30) play a major part in predicting and understanding postbuckling equilibrium and parametric resonance. For the postbuckling problem, the time dependent terms in eqn (30) should be omitted, giving

$$\left(\frac{\lambda_m}{\lambda_1} - \bar{P}_0\right) v_m + \sum_{n,s=1}^{N} \bar{b}_{mns} v_n v_s + \sum_{n,s,t=1}^{N} \bar{c}_{mnst} v_n v_s v_t = 0, \quad m = 1, 2, \ldots, N. \tag{31}$$

A state of equilibrium can be denoted by a point in the state space $(\bar{P}_0, v_1, v_2, \ldots, v_N)$ and if buckling occurs the point is called a critical point. All postbuckling equilibrium points form a curve called the postbuckling path. The tangent to the path at the critical point is referred to as the initial postbuckling slope and is important for predicting and understanding postbuckling behavior. As shown in the Appendix, the tangent is determined by the coefficient b_{111} in the following way

$$\left(\frac{\partial \bar{P}_0}{\partial v_1}, \frac{\partial \bar{P}_0}{\partial v_2}, \ldots, \frac{\partial \bar{P}_0}{\partial v_N}\right) = (b_{111}, 0, \ldots, 0) \tag{32}$$

which means that the buckling mode is in the $(\bar{P}_0, v_1)$-plane and the initial postbuckling slope is b_{111}.

It is known that for most structures it is very possible to have $b_{ijk} = 0$ for all i, j and k. In such cases the second-order derivative $\partial^2 \bar{P}_0 / \partial v_1^2$ at the critical point (i.e. the curvature of the postbuckling path there) is needed to describe the initial postbuckling behavior and the Appendix shows that

$$\frac{\partial^2 \bar{P}_0}{\partial v_1^2} = 6 c_{1111}. \tag{33}$$

Equations (32) and (33) form a complete description of the initial postbuckling behavior of a system for which eqn (30) applies.

5. PARAMETRIC RESONANCE

A time dependent compression can cause a periodic flexural vibration if the excitation frequency f is close to certain values, even for compressive loads far below the static critical value, i.e. parametric resonance occurs and the structure loses its dynamic stability. Analysis of parametric resonance involves determining the stability–instability boundaries in the parametric space (or plane) and the amplitude–frequency curves. The asymptotic method of Evan-Iwanowski (1976) is now used to solve the dynamic system of eqn (30) without repeating his detailed derivations.

Let $F_m(\bar{t}, \mathbf{v}, \mathbf{v}')$ denote the right-hand side of eqn (30), where $\mathbf{v} = (v_1, v_2, \ldots, v_N)$. Consider the following equations, which are eqn (30) with a smaller parameter ε

$$v_m'' + \Omega_m^2 v_m = \varepsilon F_m(\bar{t}, \mathbf{v}, \mathbf{v}'), \quad m = 1, 2, \ldots, N. \tag{34}$$

Once eqn (34) is solved, the solution to eqn (30) can be obtained by setting $\varepsilon = 1$. The asymptotic method seeks solutions to eqn (34) in the form

$$v_m = a_m(\tau)\cos\psi_m(\tau) + \sum_{i=1}^{M} \varepsilon^i U_{mi}(\tau, \mathbf{a}, \boldsymbol{\psi}) \tag{35}$$

$$a'_m(\tau) = \sum_{i=1}^{M} \varepsilon^i A_{mi}(\tau, \mathbf{a}, \boldsymbol{\psi}) \tag{36a}$$

$$\psi'_m(\tau) = \Omega_m + \sum_{i=1}^{M} \varepsilon^i B_{mi}(\tau, \mathbf{a}, \boldsymbol{\psi}), \quad m = 1, 2, \ldots, N \tag{36b}$$

where $\tau = \varepsilon\bar{t}$ is called slow time and $\mathbf{a}$ and $\boldsymbol{\psi}$ denote $(a_1, a_2, \ldots, a_N)$ and $(\psi_1, \psi_2, \ldots, \psi_N)$, respectively. The functions U_{mi}, A_{mi} and B_{mi} can be determined by substituting eqns (35) and (36a, b) into eqn (34), equating similar terms at the two sides of each equation and eliminating secular terms. In the following, consideration is confined to the case of $M = 1$ in order to obtain a first-order asymptotic solution, so that after setting $\varepsilon = 1$

$$v_m = a_m \cos\psi_m + U_{m1}, \quad m = 1, 2, \ldots, N \tag{37}$$

where v_m, a_m, ψ_m and U_{m1} are all functions of $\bar{t}$, but $\bar{t}$ is not written explicitly in eqn (37) and later on in this section for conciseness.

For simplicity, the first of eqn (30) is taken as a sample. The coefficients A_{11}, B_{11} and U_{11} for the first-order solution of v_1 can be obtained from the coefficients of eqn (30) by using eqns (2.23) and (2.24) of Evan-Iwanowski (1976) to give

$$A_{11} = -\tfrac{1}{2}\bar{d}_{11} \tag{38a}$$

$$B_{11} = \frac{3a_1^2}{8\Omega_1}\bar{c}_{1111} + \frac{1}{4\Omega_1}[(\bar{c}_{1122} + \bar{c}_{1221} + \bar{c}_{1212})a_2^2 + (\bar{c}_{1133} + \bar{c}_{1331} + \bar{c}_{1313})a_3^2 + , \ldots,$$
$$+ (\bar{c}_{11NN} + \bar{c}_{1NN1} + \bar{c}_{1N1N})a_N^2] \tag{38b}$$

$$U_{11} = \frac{1}{2}\bar{P}_t a_1 \left[\frac{\cos(\bar{f}\bar{t} + \psi_1)}{\bar{f}(\bar{f} + 2\Omega_1)} + \frac{\cos(\bar{f}\bar{t} - \psi_1)}{\bar{f}(\bar{f} - 2\Omega_1)}\right] + \sum_j \bar{d}_{1j}\frac{\Omega_j a_j \sin\psi_j}{\Omega_1^2 - \Omega_j^2}$$
$$- \frac{1}{2}\sum_{j,k} \bar{b}_{1jk} a_j a_k \left[\frac{\cos(\psi_j + \psi_k)}{\Omega_1^2 - (\Omega_j + \Omega_k)^2} + \frac{\cos(\psi_j - \psi)}{\Omega_1^2 - (\Omega_j - \Omega_k)^2}\right]$$
$$- \frac{1}{4}\sum_{j,k,l} \bar{c}_{1jkl} a_j a_k a_l \left[\frac{\cos(\psi_j + \psi_k + \psi_l)}{\Omega_1^2 - (\Omega_j + \Omega_k + \Omega_l)^2} + \frac{\cos(\psi_j + \psi_k - \psi_l)}{\Omega_1^2 - (\Omega_j + \Omega_k - \Omega_l)^2}\right.$$
$$\left. + \frac{\cos(\psi_j - \psi_k + \psi_l)}{\Omega_1^2 - (\Omega_j - \Omega_k + \Omega_l)^2} + \frac{\cos(\psi_j - \psi_k - \psi_l)}{\Omega_1^2 - (\Omega_j - \Omega_k - \Omega_l)^2}\right] \tag{38c}$$

where the subscripts j, k and l in the summations cover 1 through N except those which would cause a denominator to be identical to zero. Substitution of eqns (38a, b) into eqns (36a, b) with $M = 1$ and $\varepsilon = 1$ yields

$$a'_1 = -\tfrac{1}{2}\bar{d}_{11}a_1, \quad \psi'_1 = \Omega_1 + B_{11}. \tag{39a,b}$$

Equation (39a) can be solved independently for a_1 to give

$$a_1 = a_{10}\exp(-\tfrac{1}{2}\bar{d}_{11}\bar{t}) \tag{40}$$

where a_{10} is a constant determined by the initial conditions. Equation (39b) is coupled with the vibrations of other generalized displacements and can be solved together with equations similar to eqns (39a, b) for a_2, a_3, $\ldots$, a_N, without any difficulty.

Evan-Iwanowski (1976) proved that in the non-resonance case U_{11} is negligible compared to $a_1 \cos\psi_1$. Therefore, it is now omitted to give

$$v_1 = a_{10}\exp(-\tfrac{1}{2}\bar{d}_{11}\bar{t})\cos\psi_1 \tag{41}$$

which is called the zeroth-order solution.

The excitation $\bar{P}_t$ does not appear in eqn (41) because it contributed only to the small term U_{11}. Therefore, eqn (41) also forms a zeroth-order solution for nonlinear free vibration of a cylindrical shell under constant axial compression, the normalized nonlinear frequency F_1 being given by eqns (38b) and (39b) as

$$F_1 = \frac{\psi_1'}{\Omega_1} = 1 + \frac{3\bar{c}_{1111}}{8\Omega_1^2}a_1^2 + \frac{1}{4\Omega_1^2}[(\bar{c}_{1122}+\bar{c}_{1221}+\bar{c}_{1212})a_2^2 + (\bar{c}_{1133}+\bar{c}_{1331}+\bar{c}_{1313})a_3^2 +, \ldots, + (\bar{c}_{11NN}+\bar{c}_{1NN1}+\bar{c}_{1N1N})a_N^2] \tag{42}$$

which can be considered to be a generalization of the formula given by Atluri (1972).

The solution, given as eqn (41), represents a decaying process for any initial conditions, so that any disturbances to the shell which serve as initial conditions will die out exponentially. Thus, eqn (41) is called a non-resonant solution. However, the non-resonant solution is valid only when U_{11} is small. If some of the denominators in the expression for U_{11}, see eqn (38c), are close to zero the non-resonant solution of eqn (41) breaks down and the resonant case occurs.

For the resonant case, eqn (2.28) of Evan-Iwanowski (1976) gives the zeroth-order solution as

$$v_1 = a_1\cos\left(\frac{\bar{f}}{2}\bar{t}+\phi_1\right) \tag{43a}$$

$$a_1' = A_{11} - \frac{\bar{P}_t a_1}{2\bar{f}}\sin 2\phi_1 \tag{43b}$$

$$\phi_1' = \Omega_1 - \frac{\bar{f}}{2} + B_{11} - \frac{\bar{P}_t}{2\bar{f}}\cos 2\phi_1 \tag{43c}$$

where

$$\phi_1 = \psi_1 - \frac{\bar{f}}{2}\bar{t}.$$

For any initial conditions, the solution to eqns (43a–c) may converge, after a transient process, either to the stationary trivial solution or to a stationary periodic solution. The latter case means resonance, i.e. any disturbance to the shell will develop to a stationary flexural vibration.

For (43a) to be a stationary solution, it is necessary that $a_1(\bar{t})$ and $\phi_1(\bar{t})$ be constant, i.e.

$$a_1' = 0, \quad \phi_1' = 0. \tag{44}$$

Thus, from eqns (43b, c) and (38a, b), with $a_2, a_3, \ldots, a_N$ having died out because they are not in resonance, it follows that

$$a_1 = 0 \quad \text{or } a_1 = \left[\frac{8\Omega_1}{3\bar{c}_{1111}}\left(\frac{\bar{f}}{2} - \Omega_1 \pm \frac{1}{2}\left(\frac{\bar{P}_t^2}{\bar{f}^2} - \bar{d}_{11}\right)^{1/2}\right)\right]^{1/2}. \tag{45a, b}$$

The solution represented by eqn (45a) is the stationary trivial solution, and the solution given by eqns (43a) with (45b) represents the stationary periodic solution. Inspection of eqn (45b) shows that the stationary periodic solution does not always exist, and that for given Ω_1, $\bar{P}_t$ and $\bar{d}_{11}$, there are three frequencies $\bar{f}_1, \bar{f}_2$ and $\bar{f}_3$ on the $\bar{f}$-axis such that:

(a) For $\bar{f} \leqslant \bar{f}_1$ or $\bar{f} \geqslant \bar{f}_3$, a_1 is zero or imaginary, so that resonance does not occur and the system is stable, i.e. any disturbance to the system will die out exponentially.

(b) For $\bar{f}_1 < \bar{f} \leqslant \bar{f}_2$, a_1 has one non-zero real root so that one stationary periodic solution exists and the system is unstable, i.e. any disturbance to the system will develop into a stationary flexural vibration with amplitude equal to a_1.

(c) For $\bar{f}_2 < \bar{f} < \bar{f}_3$, a_1 has two non-zero real roots, and calculation shows that the system is unstable for large disturbances, i.e. large disturbances will develop into a stationary flexural vibration with its amplitude given by the larger root, whereas small disturbances will die out exponentially. Therefore, this case is called conditionally stable.

The $\bar{f}_1, \bar{f}_2$ and $\bar{f}_3$ described above are points on the $\bar{f}$-axis for any fixed $\bar{P}_t$ (and $\bar{d}_{11}$, of course). When $\bar{P}_t$ varies, the loci of these points form curves in the $(\bar{P}_t, \bar{f})$-plane, called stability–instability boundaries. The discussions in this section are about the resonance in the mode of v_1. Similar things can be done for the resonances in the modes of $v_2, v_3, \ldots, v_N$ as well.

6. NUMERICAL RESULTS

First, the method presented is checked against results from three-dimensional analyses. Figure 1 is for a rather thick cylindrical shell analyzed by Ye and Soldatos (1995), the data for which is given in the caption, noting that the inner ply is the 0° one and that the subscripts L and T denote the fiber direction and transverse direction, respectively. The curves of Fig. 1 show very good agreement of the buckling loads obtained by the present theory with the three-dimensional ones given by Ye and Soldatos (1995), especially for large E_L/E_T.

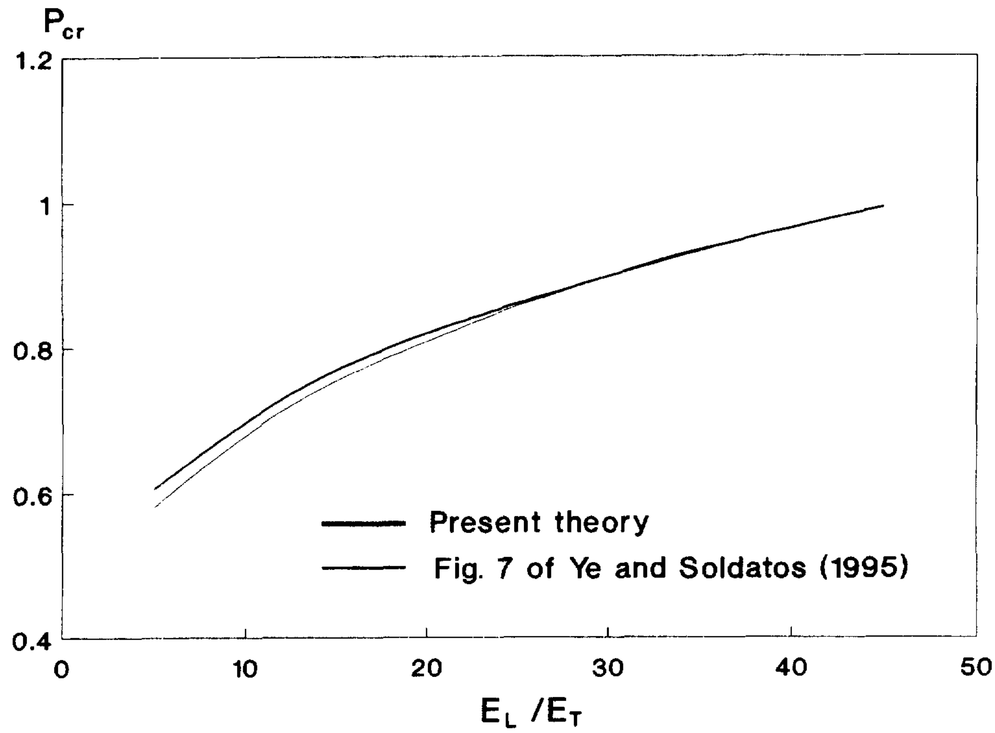

Fig. 1. Comparison of $\bar{P}_{cr}$ with the 3-D analysis of Ye and Soldatos (1995) as E_L/E_T varies. The cylinder properties are $R/T = 5$, $L/R = 5$ and [0°/90°] cross-ply with $G_{LT}/E_T = 0.6$ and $\nu_{LT} = \nu_{TT} = 0.25$.

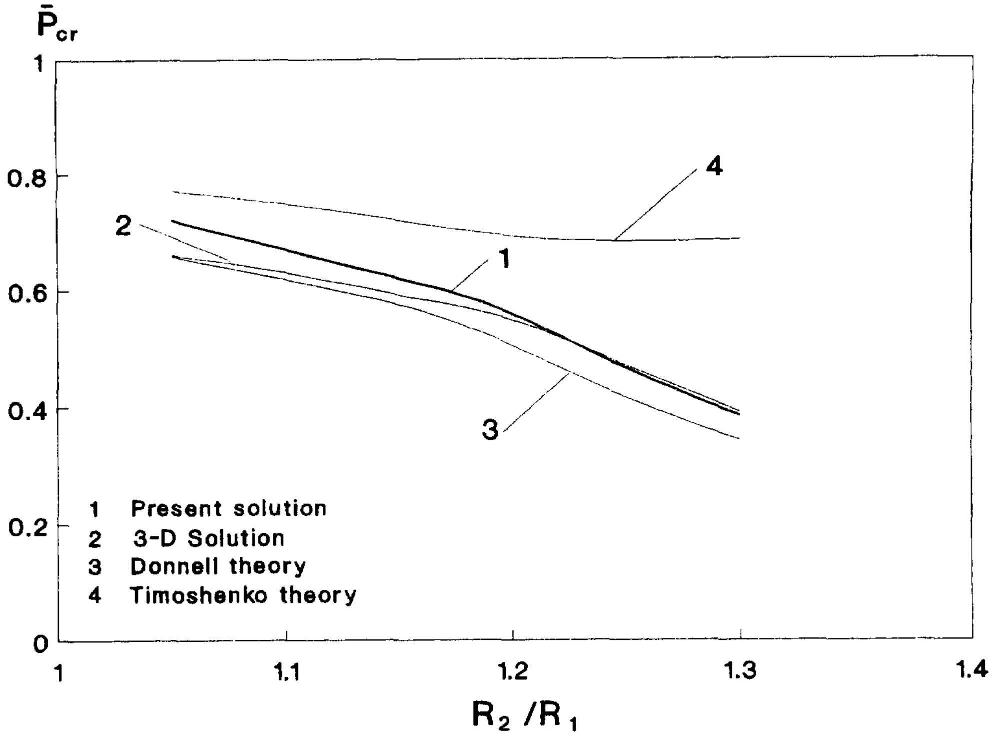

Fig. 2. Comparison of $\bar{P}_{cr}$ with the three sets of results of Kardomateas (1995) as R_2/R_1 varies. The cylinder is orthotropic with $L/R_2 = 5$, $\nu_{23} = 0.277$ and, in GPa, $E_{22} = 57$, $E_{11} = E_{33} = 14$, $G_{21} = G_{23} = 5.7$ and $G_{31} = 5.0$.

Figure 2 compares with the results of Table 1 in Kardomateas (1995), for which R_1 is the inner radius, R_2 is the outer radius and the subscripts 1, 2 and 3 denote the axial, circumferential and radial directions, respectively. The curves of Fig. 2 show that the results of the present theory are closer to those of the three-dimensional analysis of Kardomateas than are the two thin shell theories he presented, especially for thick shells.

Another numerical example is the shell analyzed by Sun (1991) based on Donnell's thin-shell theory. This is an infinitely long thin cylinder with $R/T = 165$. Its wall is a $(90°/0°/0°/90°)$ laminate with ply properties $\nu_{LT} = 0.26$ and, in GPa, $E_L = 141$, $E_T = 9.7$ and $G_{LT} = 4.1$. Sun (1991) used radial deflection w and stress function F as the basic unknowns for the analysis and assumed that they have the same buckling mode. Hence, he obtained the buckling load and initial postbuckling parameter as being dependent only on the axial and circumferential lengths of the buckling wave, regardless of the actual length of the shell, their values being $\bar{P}_{cr} = 1.32$ and $\bar{c}_{1111} = 0.079$. In the present theory, the unknown stress resultants are expanded into series, so that the parameter $\bar{c}_{1111}$ varies with the length of the shell while the critical load $\bar{P}_{cr}$ does not. Computation shows that, for large enough L/R, the axial and circumferential lengths of the buckling wave are almost fixed, and thus the buckling load is almost fully converged to $\bar{P}_{cr} = 1.34$. This value is quite close to Sun's given above because for such a thin shell Sun's Donnell shallow shell analysis is good enough within the linear sphere. However, the parameter $\bar{c}_{1111}$ given by the present theory is no longer constant, but varies from 0.060 to 0.475 as L/R increases from 0.2083 (axial wavelength) to 20. From this example it can be concluded that the nonlinear behavior, i.e. post-critical behavior, is more sensitive to the accuracy of the theory used for analysis than is the linear behavior, i.e. determination of critical loads, giving more room for refinement of its analysis.

For laminated shells, the lay-up detail of the laminate may have an appreciable influence on the buckling load and on the postbuckling behavior, especially when the wall is thick. Figure 3 is for the shell discussed in Table 2 of Iu and Chia (1988) with their SS3-type simply-supported boundary conditions. The data for the shell is given in the figure caption and the curves show that the $[90°/0°]$ lay-up gives larger buckling loads than does

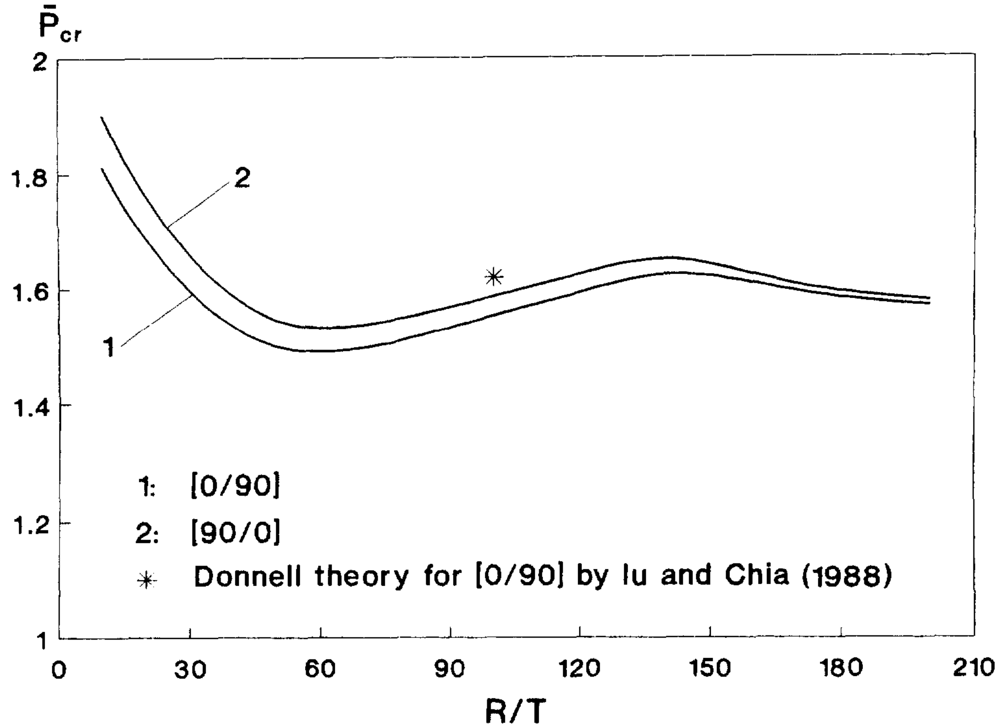

Fig. 3. Comparison of $\bar{P}_{cr}$ for [0°/90°] and [90°/0°] laminates as R/T varies. The cylinder properties are $L/R = 0.5$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.5$ and $\nu_{LT} = 0.25$.

the [0°/90°] lay-up. This is because the [90°/0°] shell has the 0° layer at its outer surface and, therefore, has more fibers in the important axial direction than does the [0°/90°] shell.

Figure 4 gives the parameter $\bar{c}_{1111}$ for the shell of Fig. 3, again with R/T varying. The curves in Fig. 4 show sudden changes in $\bar{c}_{1111}$, which are due to the changes in the number,

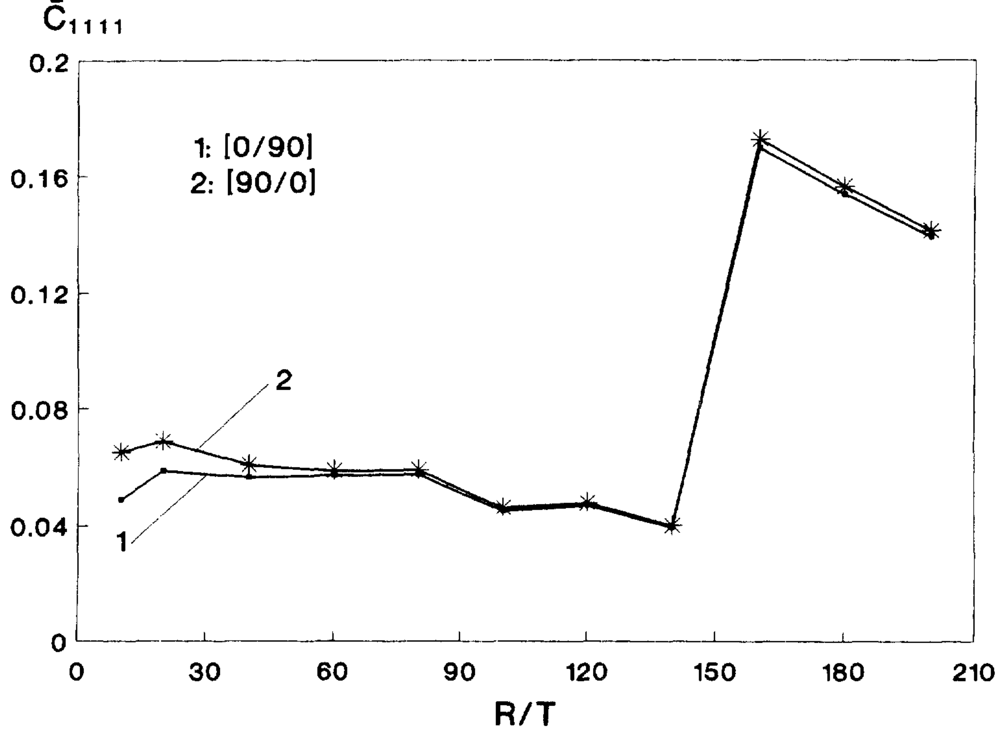

Fig. 4. Comparison of $\bar{c}_{1111}$ for [0°/90°] and [90°/0°] laminates as R/T varies. The cylinder properties are $L/R = 0.5$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.5$ and $\nu_{LT} = 0.25$.

p, of axial half waves of the buckling mode. It can be seen that the difference in $\bar{c}_{1111}$ between the [0°/90°] and [90°/0°] cases is quite substantial for thick shells, and again the [90°/0°] case gives the better performance.

Figure 5 compares stability–instability boundaries given by the present method with those of Bert and Birman (1988) for a cross-ply shell with the properties given in its caption. The load was

$$P(t) = P_t \cos ft$$

and there was no damping. The wave numbers (p, q) of the mode of resonance were fixed at the values (1, 5) used by Bert and Birman. The lines AB and AC are the boundary of the main instability region given by the present theory. The area between them is the instability region, the region above AB is conditionally stable and that below AC is stable. The lines $A'B'$ and $A'C'$ are the corresponding boundaries taken from Table 3 of Bert and Birman. The difference between the boundaries BAC and $B'A'C'$ may be partially attributed to the fact that the present theory includes the nonlinear term and the force of inertia associated with the circumferential displacement v, which tends to reduce the natural frequency. Another possible reason for this difference is that Bert and Birman used constitutive equations without the tension-bending coupling which may have an appreciable effect on the results for a [0°/90°] wall, especially when the wall is thick.

Figure 5 is for the [0°/90°] case. The corresponding stability–instability boundaries AB and AC for the [90°/0°] case in terms of the dimensionless load $\bar{P}_t$ and frequency $\bar{f}$ are identical to those of Fig. 5.

If the load parameters and damping coefficients are changed, the boundaries will change their shape and position, as shown in Fig. 6. Here, the thin lines are the boundaries AB and AC in Fig. 5, which correspond to $\bar{P}_0 = 0$ and $\bar{d}_{11} = 0$. Changing $\bar{P}_0$ from zero to 0.5 gives boundaries AB and AC as shown, whereas retaining $\bar{P}_0 = 0$ and increasing $\bar{d}_{11}$ from zero to 0.05 gives the boundaries DE, DF and DG. The region between DE and DF is

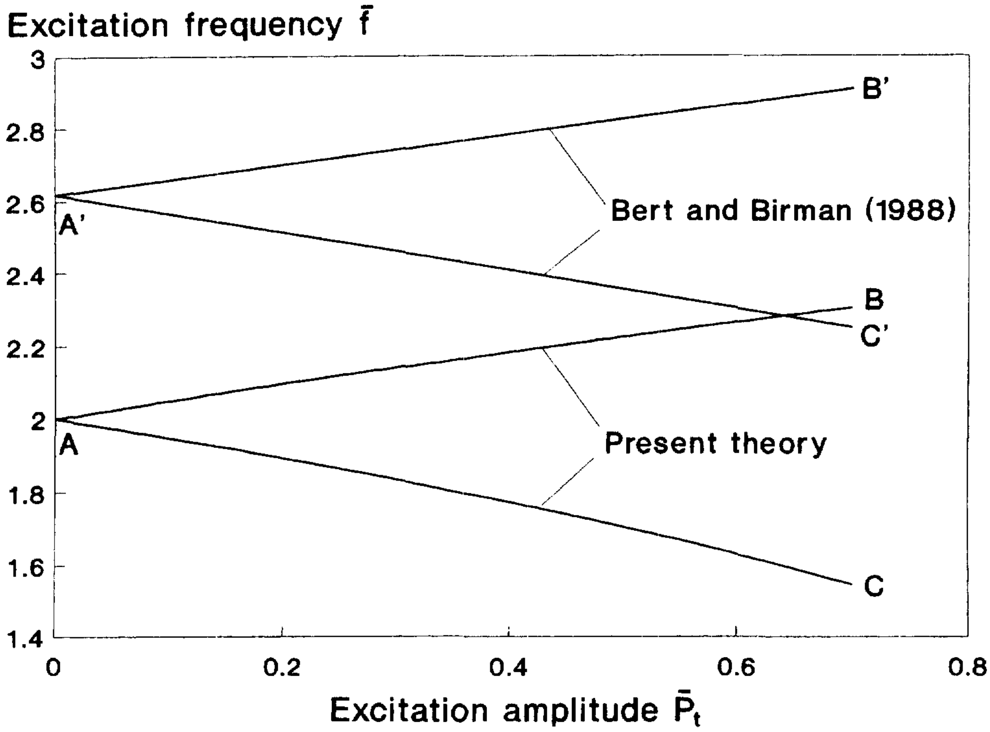

Fig. 5. Stability–instability boundaries of a [0°/90°] cross-ply cylindrical shell with $R/T = 5$, $LR = 1$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.6$, $G_{TT}/E_T = 0.5$, $v_{LT} = 0.25$, damping $\bar{d}_{11} = 0$, constant load contribution $\bar{P}_0 = 0$ and wave numbers $(p, q) = (1, 5)$.

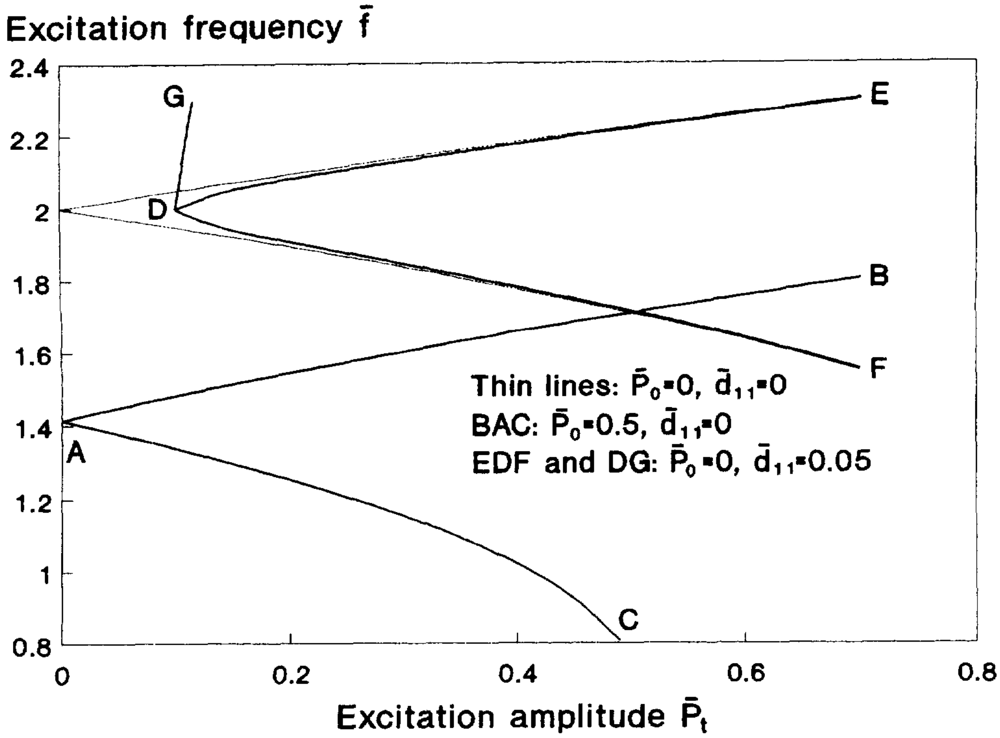

Fig. 6. Stability–instability boundaries for [0°/90°] cross-ply, cylindrical shell with $R/T = 5$, $L/R = 1$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.6$, $G_{TT}/E_T = 0.5$ and $\nu_{LT} = 0.25$.

unstable, the region between DG and DE is conditionally stable, and the remaining region is stable.

Figure 7 shows resonance curves for the shell discussed in Fig. 5, with the load and damping such that $\bar{P}_0 = 0.2$, $\bar{P}_t = 0.1$ and $\bar{d}_{11} = 0.05$ and with the wave numbers $(p, q) = (1, 2)$, which correspond to the lowest eigenfrequency. Curves 1 and 2 of Fig 7 both have two branches. The upper branch corresponds to the larger root of eqn (45b) and the lower branch corresponds to the smaller root. The intersections of the upper and lower branches with the $\bar{f}$-axis are at $\bar{f}_1$ and $\bar{f}_2$, respectively, and the abscissa at the intersection of the two branches is $\bar{f}_3$, where $\bar{f}_1, \bar{f}_2$ and $\bar{f}_3$ have the significance discussed at the end of Section 5 above. Comparison of the curves shows that the [90°/0°] lay-up is better than [0°/90°] one at reducing the amplitude of resonance.

Figure 8 demonstrates the variation of the nonlinear frequency vs amplitude curve for the shell of Fig. 1 as its length and lay-up are varied. The dimensionless frequency $F_1 (= \psi'_1/\Omega_1)$ is that of eqn (42). When $L/R = 5.0$, $F_1 = 0$ for both of the [0°/90°] and [90°/0°] cases, which means that no nonlinear effect can be observed for these cases. In fact, the basic mode of vibration of such a long shell is $p = q = 1$, i.e. the shell vibrates as a column and it is well known that the nonlinear effect for a column is very small. For shorter shells, the basic modes were $p = 1$, $q = 2$, $|V|/|W| = 0.533$ for $L/R = 1$; and $p = 1$, $q = 3$, $|V|/|W| = 0.386$ for $L/R = 0.5$. Substantial nonlinear effects on the frequency are observed from Fig. 8.

7. CONCLUSIONS

In the present paper a non-shallow thick shell theory is used to analyze unsymmetrically laminated cross-ply cylindrical shells under time dependent axial compression. Buckling loads, initial postbuckling parameters, nonlinear frequencies, stability–instability boundaries and amplitude–frequency curves, with their dependences on shell and load properties, are obtained for some numerical examples. Comparison with three-dimensional results shows good accuracy of the present theory even for very thick shells. The theory is also very good for linear and nonlinear analysis of long thick cylindrical tubes which buckle or

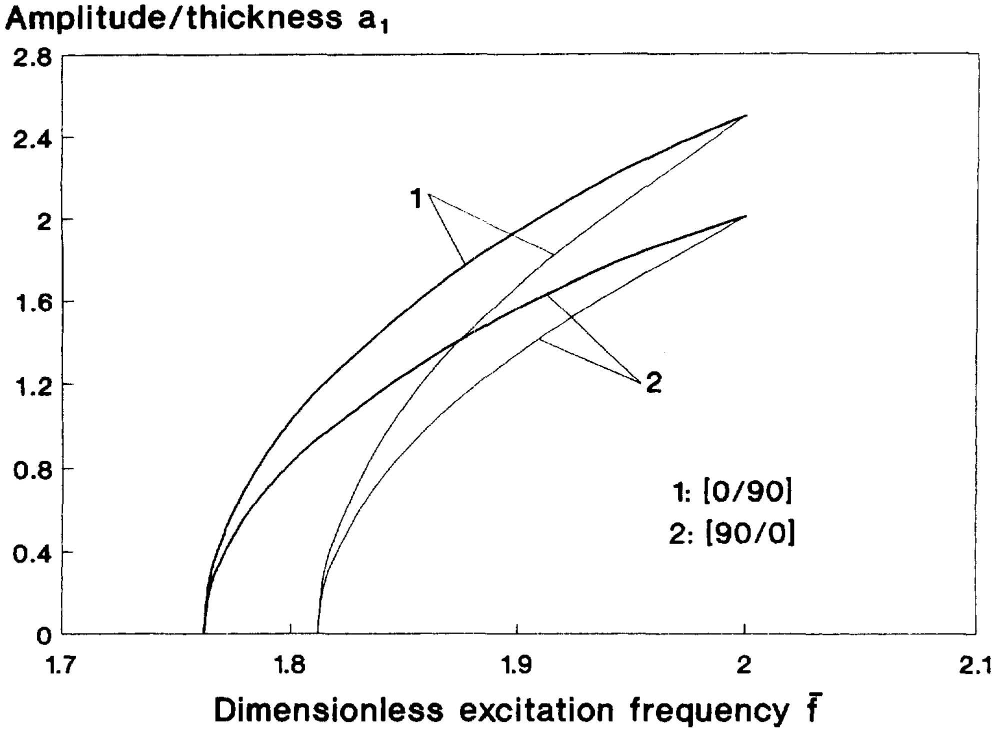

Fig. 7. Resonance curves for the basic mode $(p, q) = (1, 2)$ of a [0°/90°] or [90°/0°] cross-ply cylindrical shell with $R/T = 5$, $L/R = 1$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.6$, $G_{TT}/E_T = 0.5$ and $\nu_{LT} = 0.25$, damping $\bar{d}_{11} = 0.05$ and load components $\bar{P}_0 = 0.2$ and $\bar{P}_t = 0.1$.

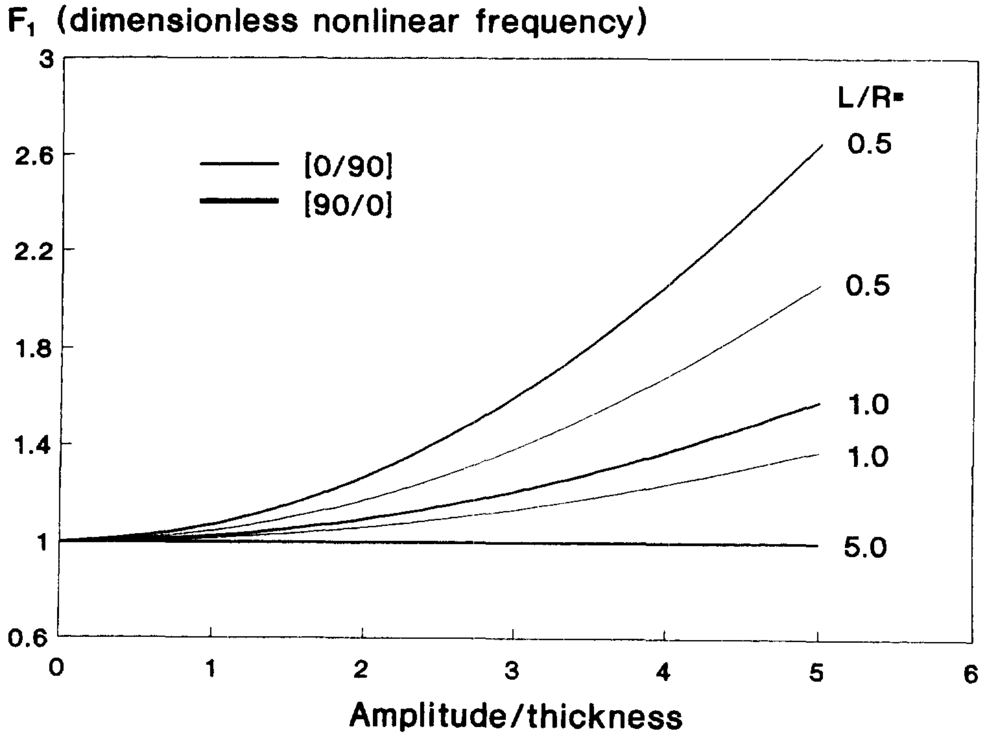

Fig. 8. Nonlinear frequencies for a [0°/90°] or [90°/0°] cross-ply cylindrical shell with $R/T = 5$, $E_L/E_T = 40$, $G_{LT}/E_T = 0.6$ and $\nu_{LT} = \nu_{TT} = 0.25$.

vibrate as columns. Numerical examples also show the effect of the lay-up details on the critical and post-critical behaviors for thick unsymmetrically laminated shells.

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APPENDIX

The derivation given below is mainly based on Rik (1979). Consider an equilibrium governed by the eqns

$$f_m(\bar{P}_0, \mathbf{v}) = 0, \quad m = 1, 2, \ldots, N \tag{A1}$$

with the condition

$$f_{m,n} = f_{n,m} \tag{A2}$$

where $\mathbf{v}$ denotes $(v_1, v_2, \ldots, v_N)$, and $f_{m,n} = \partial f_m/\partial v_n$. Equation (A1) determines the path of equilibrium in the $(N+1)$-dimensional space S composed of $\bar{P}_0$ and $\mathbf{v}$. The theorem of existence of implicit functions states that if

$$\det|f_{m,n}(\bar{P}_0, \mathbf{v})| \neq 0 \tag{A3}$$

at a point in space S, then eqn (A1) uniquely determines the v_n $(n = 1, 2, \ldots, N)$ as single-valued functions of $\bar{P}_0$ in a neighborhood of that point. A point where

$$\det|f_{m,n}(\bar{P}_0, \mathbf{v})| = 0 \tag{A4}$$

is then called a point of singularity. A point of singularity may be a turning point (limit point), where the tangent to the path of equilibrium is perpendicular to the direction of $\bar{P}_0$, or a bifurcation point, where the path bifurcates and has more than one tangent. The bifurcation is of the most interest.

For eqn (31), the point $\bar{P}_0 = 1$ on the $\bar{P}_0$-axis is a bifurcation, called the critical point, where the path bifurcates into two branches. One is the fundamental path with all $v_m = 0$ and the other is the postbuckling path. The slope and the curvature of the postbuckling path at the critical point are to be calculated from the coefficients of eqn (31).

Differentiating eqn (A1) with respect to the length s of the posbuckling path gives

$$\frac{\mathrm{d}}{\mathrm{d}s} f_m(\bar{P}_0, \mathbf{v}) = \frac{\partial f_m}{\partial \bar{P}_0}\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s} + \sum_{i=1}^{N} f_{m,i}\frac{\mathrm{d}v_i}{\mathrm{d}s} \tag{A5}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} f_m(\bar{P}_0, \mathbf{v}) = \frac{\partial^2 f_m}{\partial s\,\partial \bar{P}_0}\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s} + \frac{\partial f_m}{\partial \bar{P}_0}\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} + \sum_{i=1}^{N}\left(\frac{\partial f_{m,i}}{\partial s}\frac{\mathrm{d}v_i}{\mathrm{d}s} + f_{m,i}\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2}\right) = 0, \quad m = 1, 2, \ldots, N \tag{A6}$$

which are valid at all points on the postbuckling path. Let $(\mathrm{d}\bar{P}_0/\mathrm{d}s, \mathrm{d}\mathbf{v}/\mathrm{d}s)_1$, and $(\mathrm{d}\bar{P}_0/\mathrm{d}s, \mathrm{d}\mathbf{v}/\mathrm{d}s)_2$, be the tangential vectors to the fundamental path and the postbuckling path, respectively, at the bifurcation, and $\mathbf{v}^*$ be the buckling mode. Rik (1979) shows that there is a linear relation between the tangential vectors and the buckling mode, which can be written in the form

$$\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\right)_2 = \alpha\beta\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\right)_1 \tag{A7}$$

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right)_2 = \alpha\left(\beta\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right)_1 + \mathbf{v}^*\right) \tag{A8}$$

and proves that

$$\beta = -\frac{1}{2}\left(\sum_{i,j,k=1}^{N} f_{i,jk} v_i^* v_j^* v_k^*\right) \bigg/ \left(\sum_{i,j,k=1}^{N} f_{i,jk} v_i^* v_j^* \left(\frac{\mathrm{d}v_k}{\mathrm{d}s}\right)_1 + \sum_{i,j=1}^{N} \frac{\partial f_{i,j}}{\partial \bar{P}_0}\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s} v_i^* v_j^*\right) \tag{A9}$$

$$\alpha = \left(\beta^2 + 2\beta\sum_{i=1}^{N} v_i^*\left(\frac{\mathrm{d}v_i}{\mathrm{d}s}\right)_1 + 1\right)^{-0.5}. \tag{A10}$$

For eqn (31)

$$(\bar{P}_0^c, \mathbf{v}^c) = (1, 0, \ldots, 0), \quad \mathbf{v}^* = (1, 0, \ldots, 0), \quad \left(\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\right)_1, \left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right)_1\right) = (1, 0, \ldots, 0) \tag{A11}$$

and, therefore

$$\beta = \frac{f_{1,11}}{2} = \bar{b}_{111}, \quad \alpha = \frac{1}{\sqrt{1+\beta^2}} = \frac{1}{\sqrt{1+\bar{b}_{111}^2}} \tag{A12}$$

$$\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right)_2 = \frac{1}{\sqrt{1+\bar{b}_{111}^2}}(\bar{b}_{111}, 1, 0, \ldots, 0). \tag{A13}$$

From eqn (A13) the initial postbuckling slope can be obtained as

$$\frac{\partial \bar{P}_0}{\partial v_1} = \frac{\partial \bar{P}_0}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}v_1} = \bar{b}_{111}. \tag{A14}$$

For most structures, it is very possible to have $b_{ijk} = 0$ for all i, j and k, so that the second-order derivatives are needed. These can be obtained from eqn (A6) under the condition of normalization

$$\left(\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\right)^2 + \sum_{i=1}^{N}\left(\frac{\mathrm{d}v_i}{\mathrm{d}s}\right)^2 = 1. \tag{A15}$$

Equation (A15) can be differentiated to give

$$\frac{\mathrm{d}\bar{P}_0}{\mathrm{d}s}\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} + \sum_{i=1}^{N}\frac{\mathrm{d}v_i}{\mathrm{d}s}\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2} = 0. \tag{A16}$$

Combination of eqns (A13), (A16) and the condition $b_{ijk} = 0$ gives

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}s^2} = 0. \tag{A17}$$

Then eqn (A6) yields

$$\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} = 6\bar{c}_{1111} \tag{A18}$$

$$\frac{\mathrm{d}^2 v_i}{\mathrm{d}s^2} = 0, \quad i = 2, 3, \ldots, N. \tag{A19}$$

By making use of eqns (A13) and (A17) it can be shown that

$$\frac{\mathrm{d}^2\bar{P}_0}{\mathrm{d}s^2} = \frac{\partial^2\bar{P}_0}{\partial v_1^2}. \tag{A20}$$

Hence eqn (A18) yields

$$\frac{\partial^2\bar{P}_0}{\partial v_1^2} = 6\bar{c}_{1111}. \tag{A21}$$